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HARNACK ESTIMATE FOR THE MEAN CURVATURE FLOW

RICHARD S. HAMILTON

1. The result

We consider the evolution of a hypersurface M^n in Euclidean space R^{n+1} by its mean curvature. This was first studied by Huisken [3]. In this flow each point Y on M moves in the direction of the unit normal vector N with velocity equal to the mean curvature H, the trace of the second fundamental form $H(V, V)$ over the tangent vectors V. We confine our attention to solutions which are smooth and either compact, or else are complete with bounded second fundamental form.

1.1. Main Theorem A. *For any weakly convex solution to the mean curvature flow for $t > 0$ we have*

$$\frac{\partial H}{\partial t} + \frac{1}{2t}H + 2DH(V) + H(V, V) \geq 0$$

for all tangent vectors V.

This is the differential Harnack inequality for the mean curvature flow. As usual (see Li and Yau [4]) we can integrate over paths in space-time to get an integral Harnack inequality.

1.2. Corollary. *For any weakly convex solution to the mean curvature flow for $t > 0$ we have*

$$H(Y_2, t_2) \geq \sqrt{t_1/t_2}e^{-\Delta/4}H(Y_1, t_1)$$

for any two points Y_1 and Y_2 on the evolving surface at times t_1 and t_2 with $0 < t_1 < t_2$, where

$$\Delta = \inf \int \left|\frac{dY}{dt}\right|_M^2 dt$$

is the infimum over all paths $Y(t)$ remaining on the surface at time t with $Y = Y_1$ at $t = t_1$ and $Y = Y_2$ at $t = t_2$, dY/dt is the velocity vector

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of the path, and $|dY/dt|_M$ *is the length of its component tangent to the surface* M. *In particular*

$$\Delta \leq d(Y_1, \widehat{Y}_2, t_1)^2/(t_2 - t_1),$$

where $d(Y_1, \widehat{Y}_2, t_1)$ *is the distance along the surface at time* t_1 *between* Y_1 *and the point* $\widehat{Y}_2$ *which evolves normally to* Y_2 *at time* t_2.

We say a solution is eternal if it is defined for $-\infty < t < \infty$. One interesting class of eternal solutions is the translating solitons. These are surfaces which evolve by translating in space with a constant velocity. The "grim reaper" curve $y = ln \sec x$ in the plane is one example. In space there is a convex rotationally symmetric translating soliton, which unlike the grim reaper opens to infinite width. There is another complete rotationally invariant translating soliton which is not convex.

Eternal solutions arise as limits of dilations (in space-time) of slowly forming singularities. Therefore it is important to classify them. When we take such a limit, we can always put the maximum of the extrinsic curvature at the origin at time zero. Therefore the following result is useful for convex limits.

1.3. Main Theorem B. *Any strictly convex eternal solution to the mean curvature flow where the mean curvature assumes its maximum value at a point in space-time must be translating soliton.*

The proof is a consequence of the Harnack estimate above for the mean curvature flow and the strong maximum principle. Analogous results for the Ricci flow occur in [1] and [2].

2. Notation

We begin by fixing our notation. We suppose the surface M is parametrized locally by $X = \{x^i\}$ in R^n and t. Then a point $Y = \{y^\alpha\}$ in R^{n+1} on M is given locally by

$$y^\alpha = F^\alpha(x^i, t).$$

The tangent space is parametrized as the image of the linear map $DY = \{D_i y^\alpha\}$ where $D_i y^\alpha = \partial y^\alpha / \partial x^i$. If $I_{\alpha\beta}$ denotes the Euclidean metric on R^{n+1}, then the induced metric $G = \{g_{ij}\}$ on M is given by

$$g_{ij} = I_{\alpha\beta} D_i y^\alpha D_j y^\beta.$$

The unit normal $N = \{N^\alpha\}$ is defined by

$$I_{\alpha\beta} N^\alpha N^\beta = 1 \quad \text{and} \quad I_{\alpha\beta} N^\alpha D_i y^\beta = 0$$

and a choice of sign; on a convex surface we take N to point inward. The metric $G = \{g_{ij}\}$ induces a Levi-Civita connection $\Gamma = \{\Gamma_{ij}^k\}$ on M, which allows us to take covariant derivatives $D = \{D_i\}$ of tensors on M. Regarding the $D_j y^\alpha$ as a collection of one-forms on M, their covariant derivatives are

$$D_i D_j y^\alpha = H_{ij} N^\alpha ,$$

where $A = \{H_{ij}\}$ is the second fundamental form on M. Its trace $H = g^{ij} H_{ij}$ is the mean curvature.

The mean curvature flow is given by the equation

$$\partial Y/\partial t = HN ,$$

which in local coordinates becomes

$$\partial y^\alpha/\partial t = HN^\alpha .$$

Since the Laplacian is the trace of the second covariant derivative,

$$\Delta y^\alpha = g^{ij} D_i D_j y^\alpha = HN^\alpha ,$$

and we can also write the mean curvature flow as

$$\partial Y/\partial t = \Delta Y ,$$

which shows it is the heat equation for a submanifold.

Rather than work in local coordinates, it is more geometric to work in an orthonormal frame $F = \{F_a^i\}$, where the vectors $F_1, \cdots, F_n$ are tangent to M. To keep the vectors orthonormal and tangent to M we must let them evolve by the equation

$$\frac{\partial}{\partial t} F_a^i = g^{ij} H H_{jk} F_a^k .$$

Then we always have

$$g_{ij} F_a^i F_b^j = I_{ab} ,$$

where I_{ab} is the identity. We can then write the components of any tensor in terms of the frame, for example $H_{ab} = H(F_a, F_b)$ or in local coordinates, $H_{ab} = H_{ij} F_a^i F_b^j$. Indices $a, b, c, \cdots$ on a tensor always refer to the frame, while $i, j, k, \cdots$ refer to the local coordinates. We can take covariant derivatives also in the frame coordinates, so that for example $D_a V_b = F_a^i F_b^j D_i V_j$. Then the operators D_a become vector fields on the orthonormal frame bundle. We also have the time-like vector field D_t on the frame bundle, which differentiates in the direction of the moving frame. Thus in local coordinates

$$D_t V_a = \left\{ \frac{\partial}{\partial t} V_k + g^{ij} H H_{jk} V_i \right\} F_a^k .$$

We can then compute the commutator of D_t and D_a; a tedious calculation shows that we get the following formula.

2.1. Formula.

$$D_t D_a V_b - D_a D_t V_b = H H_{ac} D_c V_b + (H_{ac} D_b H - H_{ab} D_c H) V_c.$$

Fortunately we only use the following Corollary, which is easier to deduce on its own. We leave the details as an exercise.

2.2. Corollary. *For any function* f

$$(D_t - \Delta) D_a f = D_a (D_t - \Delta) f + H_{ac} H_{cd} D_d f$$

and

$$(D_t - \Delta)\Delta f = \Delta(D_t - \Delta) f + 2 H H_{ab} D_a D_b f + 2 H_{ab} D_a H D_b f.$$

The evolution of the second fundamental form in the moving frame is particularly simple.

2.3. Theorem. *The tensor* H_{ab} *evolves by the formula*

$$D_t H_{ab} = \Delta H_{ab} + |H_{cd}|^2 H_{ab},$$

where $\Delta H_{ab} = D_c D_c H_{ab}$ *is the trace of its second covariation derivative, and* $|H_{cd}|^2 = H_{cd} H_{cd}$ *is the length squared of the second fundamental form.*

This result occurs in [3], except some terms disappear in the moving frame.

3. Solitons

Harnack estimates are closely related to soliton solutions. A soliton is a solution which moves under a one-parameter subgroup of the symmetry group of the equation. For the mean curvature flow the symmetries are translation, rotation, and homothetic stretching. For our current purposes it will suffice to look at the translating solitons.

Suppose a solution to the mean curvature flow translates in the direction of the constant vector $T = \{T^\alpha\}$. Let $V = \{V^\alpha\}$ be the tangential part of T. Then the normal component must be HN^α to solve the mean curvature flow. In local coordinates

$$V^\alpha = V^i D_i y^\alpha,$$

where $\{V^i\}$ is a tangent vector on M. We also consider the covector $V_i = g_{ij} V^j$. If we take the equation $V^\alpha + HN^\alpha = T^\alpha$ and covariant differentiate it, and use the fact that $D_i T^\alpha = 0$, then equating tangential and normal components we find that

$$D_i V_j = H H_{ij} \quad \text{and} \quad D_i H + H_{ij} V_j = 0.$$

The first equation just says that the metric evolves by the Lie derivative with respect to V, since

$$(\mathscr{L}_V g)_{ij} = D_i V_j + D_j V_i$$

and for the mean curvature flow

$$\frac{\partial}{\partial t} g_{ij} = -2 H H_{ij}.$$

Moreover since the first equation implies that $D_i V_j = D_j V_i$, it follows that there is a function f with $D_i f = V_i$. The fact that such an f can be found globally on M follows from the observation that any complete surface $M^n \subseteq R^{n+1}$ for $n \geq 2$ is simply connected. Since the vector field is a gradient, we see that translating solitons are gradient solitons. (We will show the converse later.)

So our translating soliton satisfies the equation

$$D_i D_j f = H H_{ij}.$$

We can deduce the other equation above as follows. Take the covariant derivative of each side to get

$$D_i D_j D_k f = H_{jk} D_i H + H D_i H_{jk},$$

switch i and j and subtract, and use the Codazzi identity

$$D_i H_{jk} = D_j H_{ik}$$

to conclude that

$$R_{ijkl} D_l f = H_{jk} D_i H - H_{ik} D_j H.$$

From the Gauss curvature equation we have

$$R_{ijkl} = H_{ik} H_{jl} - H_{il} H_{jk},$$

and it follows that

$$H_{jk}(D_i H + H_{il} D_l f) = H_{ik}(D_j H + H_{jl} D_l f).$$

Let us put

$$X_i = D_i H + H_{ij} V_j.$$

Then

$$H_{jk} X_i = H_{ik} X_j.$$

Take the trace on j and k to obtain

$$(Hg_{ij} - H_{ij})X_j = 0.$$

If the surface is strictly convex, then the matrix $Hg_{ij} - H_{ij}$ is strictly positive, so $X_j = 0$. If the surface has some zeros in the second fundamental form but solves the mean curvature flow, it splits as a product of a strictly convex piece with a trivial flat direction, and hence the result still holds.

There is another relation that holds on a translating soliton.

3.1. Lemma. *On a translating soliton* $D_t H_{ab} + V_c D_c H_{ab} = 0$.

Proof. We now work in the evolving frame. We have

$$D_a H + H_{ab} V_b = 0.$$

Differentiating and using

$$D_a V_b = H H_{ab}$$

we get

$$D_a D_b H + H H_{ac} H_{bc} + V_c D_c H_{ab} = 0.$$

However,

$$D_t H_{ab} = \Delta H_{ab} + |H_{cd}|^2 H_{ab} = D_a D_b H + H H_{ac} H_{bc}$$

follows from the Gauss and Codazzi identities, and this proves the lemma.

We now look for a quadratic expression in V, which vanishes on the soliton, and whose first variation in V also vanishes on the soliton. This is because the Harnack inequality must become an equality on the translating soliton. Taking the trace of the relation in Lemma 3.1 we get

$$D_t H + V_a D_a H = 0.$$

Moreover we have

$$D_a H + H_{ab} V_b = 0$$

which yields

$$V_a D_a H + H_{ab} V_a V_b = 0.$$

Adding we get the following.

3.2. Lemma. *On a translating soliton*

$$D_t H + 2DH(V) + H(V, V) = 0.$$

This is the basic expression in the Harnack inequality. But in the theorem we use a slightly different form, which includes the term $(1/2t)H$. This is because for solutions on $t > 0$ we should really consider homothetically expanding solitons. They differ from the translating ones by terms that are basically lower order with a $1/t$ in them. It is usually easier to look at the steady solitons first, and add the $1/t$ term later.

4. The computation

We need to find how the basic Harnack expression evolves under the flow. To this end we define the following basic quantities, which all vanish on a translating soliton.

4.1. Definition. We let

$$\begin{aligned} X_a &= D_a H + H_{ab} V_b, \\ Y_{ab} &= D_a V_b - H H_{ab}, \\ Z &= D_t H + 2 V_a D_a H + H_{ab} V_a V_b, \\ W_{ab} &= D_t H_{ab} + V_c D_c H_{ab}, \\ W &= D_t H + V_c D_c H, \\ U_a &= (D_t - \Delta) V_a + H_{ab} D_b H, \end{aligned}$$

all of which vanish on a translating soliton for the induced V.

4.2. Computation. *For any solution of the mean curvature flow and any vector field V we have*

$$(D_t - \Delta) Z = |H_{ab}|^2 Z + 2 X_a U_a - 2 H_{bc} Y_{ab} Y_{ac} - 4 Y_{ab} W_{ab}.$$

Proof. The computation is long but straightforward using the preceding rules. We summarize the relevant parts.

First we compute

$$(D_t - \Delta) D_a H = D_a (D_t - \Delta) H + H_{ab} H_{bc} D_c H,$$

which gives us

$$(D_t - \Delta) D_a H = |H_{bc}|^2 D_a H + 2 H H_{bc} D_a H_{bc} + H_{ab} H_{bc} D_c H.$$

From this we get

$$(D_t - \Delta) X_a = |H_{bc}|^2 X_a - 2 Y_{bc} D_a H_{bc} + H_{ab} U_b.$$

Note that since X vanishes on a translating soliton, so does its evolution. This helps to avoid errors in the computation.

Next we find that

$$\begin{aligned} (D_t - \Delta) \Delta H = \Delta (D_t - \Delta) H &+ 2 H H_{ab} D_a D_b H \\ &+ 2 H_{ab} D_a H D_b H, \end{aligned}$$

and since $D_t H = \Delta H + |H_{ab}|^2 H$, we get

$$\begin{aligned} (D_t - \Delta) D_t H = |H_{ab}|^2 D_t H &+ 4 H H_{ab} D_t H_{ab} \\ &+ 2 H_{ab} D_a H D_b H - 2 H^2 H_{ab} H_{ac} H_{bc}, \end{aligned}$$

which leads us to the formula

$$(D_t - \Delta)W = |H_{ab}|^2 W + (4HH_{ab} - 2D_a V_b)W_{ab} + H_{ab}D_a H \cdot X_b + D_a H \cdot V_a + 2[HH_{ac}H_{bc} + V_c D_c H_{ab}]Y_{ab}.$$

Finally we use $Z = W + V_a X_a$ to compute the evolution of Z, giving the result stated. Note that since Z vanishes on a translating soliton but is positive on other solutions, in some sense Z vanishes quadratically on a translating soliton, so all the terms in its evolution do also.

It remains to add the correct factors of $1/t$. To this end we make the following definition.

4.3. Definition. We let

$$\widetilde{X}_a = X_a, \ \widetilde{Y}_{ab} = Y_{ab} - \frac{1}{2t}g_{ab},$$
$$\widetilde{Z} = Z + \frac{1}{2t}H, \widetilde{W}_{ab} = W_{ab} + \frac{1}{2t}H_{ab},$$
$$\widetilde{W} = W + \frac{1}{2t}H, \ \widetilde{U}_a = U_a + \frac{1}{t}V_a.$$

4.4. Corollary. *For any solution to the mean curvature flow on $t > 0$ and for any vector field V we have*

$$(D_t - \Delta)\widetilde{Z} = \left(|H_{ab}|^2 - \frac{2}{t}\right)\widetilde{Z} + 2\widetilde{X}_a\widetilde{U}_a - 2H_{bc}\widetilde{Y}_{ab}\widetilde{Y}_{ac} - 4\widetilde{Y}_{ab}\widetilde{W}_{ab}.$$

5. The maximum principle

It is now easy to derive the differential Harnack inequality from the maximum principle, based on the previous calculation. We only need to prove that $\widetilde{Z} \geq 0$ for any choice of V. Suppose then that $\widetilde{Z} = 0$ at some point for some choice of V there. We can extend V to a vector field in space-time so that at the original point $\widetilde{Y}_{ab} = 0$; and if we want we can make $\widetilde{U}_a = 0$ there also, although this is unnecessary, for the first variation of $\widetilde{Z}$ with respect to V_a is $2\widetilde{X}_a$, and hence when $\widetilde{Z}$ is first zero we must have $\widetilde{X}_a = 0$ also. Then we have $(D_t - \Delta)\widetilde{Z} = 0$, so $\widetilde{Z}$ cannot become negative. A formal proof requires sticking in some small perturbation $\varepsilon > 0$, or for the noncompact case sticking in a small positive function with bounded gradient which goes to infinity as a point goes to infinity. This can be done analogously to the case for the Ricci flow in [1], so we omit repeating the details.

6. The path integral

We can now obtain the integrated version of the Harnack estimate by integrating along paths in space-time as in the work of Li and Yau [4]. Along any path

$$Y(t) = F(X(t), t)$$

we have

$$\frac{dH}{dt} = D_t H + DH\left(\frac{dX}{dt}\right),$$

and hence from the Harnack estimate we obtain, taking $V = \frac{1}{2}(dX/dt)$,

$$\frac{dH}{dt} \geq -\frac{1}{4}H\left(\frac{dX}{dt}, \frac{dX}{dt}\right) - \frac{1}{2t}H.$$

Now for a convex surface,

$$H(V, V) \leq H|V|^2$$

and hence

$$\frac{d}{dt}\log H \geq -\frac{1}{4}\left|\frac{dX}{dt}\right|^2 - \frac{1}{2t}.$$

Note that dX/dt is the tangential component of dY/dt, so that

$$\log H(Y_2, t_2)/H(Y_1, t_1) \geq -\tfrac{1}{2}\log(t_2/t_1) - \tfrac{1}{4}\Delta,$$

where

$$\Delta = \int \left|\frac{dY}{dt}\right|_M^2 dt,$$

and the result follows by exponentiating.

7. The strong maximum principle

Now we turn our attention to eternal solutions and prove Main Theorem B. If the solution is weakly convex and eternal, then since it exists for time $t > -\infty$ the Harnack estimate implies that $Z \geq 0$ everywhere for all V.

We now assume our solution is eternal and strictly convex, so that $H_{ab} > 0$. Then the quadratic form Z can vanish in at most one direction V. If the mean curvature H assumes its maximum at a point in space-time, then at that point $D_t H = 0$ and $D_a H = 0$, so $Z = 0$ in the direction $V = 0$. Thus the strong maximum principle implies that there must be exactly one V at each point in space-time where $Z = 0$, and V will vary smoothly. A formal proof can be given from the following lemma.

7.1. Lemma. *If $F \geq 0$ is a weakly positive function on M satisfying $(D_t - \Delta)F = 0$ and if $Z \geq F$ at time α for all V, then $Z \geq F$ at all subsequent times for all V.*

Proof. We use the maximum principle. To make the argument rigorous we need two small functions φ and ψ, which we let go to zero in the limit. We need φ to depend on space and $\varphi \to \infty$ as we go off in space, while ψ will depend on time only. Put

$$\widehat{Z} = Z - F + \varphi + \psi|V|^2.$$

If we can show $\widehat{Z} > 0$ for suitable small φ and ψ, we will be done. Otherwise suppose we look at the first time where $\widehat{Z} = 0$ at some point in some direction V. We extend V to a vector field as before to make $U_a = 0$ and $Y_{ab} = 0$ there. We then have at this point

$$(D_t - \Delta)\widehat{Z} = |H_{ab}|^2(F - \varphi - \psi|V|^2) + (D_t - \Delta)\varphi + (D_t\psi)|V|^2.$$

Now $F \geq 0$ and $|H_{ab}|^2 \leq C$. Since $D_t\widehat{Z} \leq 0$ and $\Delta\widehat{Z} \geq 0$ at this point, we get a contradiction if $(D_t - \Delta)\varphi > C\varphi$ and $D_t\psi \geq C\psi$. The latter is easily done for small ψ. The former requires a little care, because we need to have $\varphi \to \infty$ in the space direction to guarantee the existence of a minimum of $\widehat{Z}$ at a finite point. However, it is not hard any more to cobble such a φ. We can make it small by multiplying by ε. This proves the lemma.

We apply it as follows. The quadratic Z vanishes for at most one V. If there is ever a point where $Z > 0$ for all V, we can find an $F \geq 0$ with support in a neighborhood of that point so that $F > 0$ at the point and $Z \geq F$ for all V at all points at that time. Then we can evolve F by the heat equation, and after any positive time has elapsed, F is strictly positive everywhere by the strong maximum principle for functions. Hence subsequently Z has no zeros.

But Z has a zero at $V = 0$ at the point in space-time where H assumes its maximum. Therefore prior to that Z has a zero vector V everywhere. Since $Z \geq 0$ and $H_{ab} > 0$, the zero vector V is unique and varies smoothly. To be specific $V_a = -H_{ab}^{-1} D_b H$.

Now by real analyticity we get $Z = 0$ at this V for later times as well. (This is not strictly necessary to complete the argument, but it is reassuring.) We can even conclude more. Start at a point in the direction V where $Z = 0$, and extend V in a neighborhood so that

$$V_a = -X_a \quad \text{and} \quad Y_{ab} = -W_{ad}H_{bd}^{-1}$$

at that point. Then at the point (recall $Z = 0$)

$$(D_t - \Delta)Z = 2|X_a|^2 + 2H_{bc}^{-1}W_{ab}W_{ac}.$$

Therefore unless $X_a = 0$ and $W_{ab} = 0$ we would conclude $D_t Z > 0$, and Z would have been strictly negative at that point a little earlier. But this can never be. Therefore $X_a = 0$ and $W_{ab} = 0$ for this choice of V_a. The fact that $X_a = 0$ follows from the fact $Z = 0$, but $W_{ab} = 0$ is new information.

8. Translating solitons

The rest of the proof of the Main Theorem B now reduces to the following local computation.

8.1. Theorem. *If we have a solution to the mean curvature flow which is strictly convex, and a vector field V_a satisfying*

$$D_a H + H_{ab} V_b = 0$$

and

$$D_t H_{ab} + V_c D_c H_{ab} = 0,$$

then the solution is a translating soliton.

Proof. To begin we differentiate the first equation. This gives

$$D_a D_b H + V_c D_c H_{ab} + H_{bc} D_a V_c = 0.$$

Then we use

$$D_t H_{ab} = D_a D_b H + H H_{ac} H_{bc}$$

to conclude that

$$H_{bc} D_a V_c = H H_{ac} H_{bc},$$

and since $H_{bc} > 0$ is invertible, we get

$$D_a V_c = H H_{ac}.$$

It follows as before that $V_c = D_c f$ for some globally defined f.

Consider the vector

$$T^\alpha = g^{ij} V_i D_j y^\alpha + H N^\alpha.$$

Its covariant derivative is

$$D_i T^\alpha = g^{jk}(D_i V_j - H H_{ij}) D_k y^\alpha + (D_i H + g^{jk} H_{ij} V_k) N^\alpha,$$

and thus our equations show that $D_i T^\alpha = 0$. Hence T^α is a constant vector field. Since its normal component is always H, the motion of the surface must be translation in the direction T.

References

[1] R. S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Differential Geometry **37** (1993) 225–243.
[2] ——, *Eternal solutions to the Ricci flow*, J. Differential Geometry **38** (1993) 1–11.
[3] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geometry **20** (1984) 237–266.
[4] P. Li & S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1968) 153–201.

UNIVERSITY OF CALIFORNIA, SAN DIEGO